Moutard type transform for matrix generalized analytic functions and gauge transforms

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Considerable progress in the theory of Darboux-Moutard type transforms for two-dimensional linear differential systems with applications to geometry, spectral theory, and soliton equations has been achieved recently, see, e.g., [1, 2, 3, 4]. In the present note we derive such a transformation for the matrix generalized function system

$$\partial_{\bar{z}}\Psi + A\Psi + B\bar{\Psi} = 0, \tag{1}$$

where $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$, the coefficients A and B and solutions Ψ are $(N \times N)$-matrix functions on D, with D an open simply connected domain in $\mathbb{C}$. In particular, this generalizes the transform for $N = 1$ found in [4] with $A = 0$. In addition, we show that the Moutard type transform for system (1) with $B = 0$ is equivalent to a gauge transform for the connection $\nabla_{\bar{z}} = \partial_{\bar{z}} + A$. In turn, our studies show that the Moutard type transform for system (1) with $A = 0$ can be treated as a proper analog of the aforementioned gauge transform.

As for $N = 1$, system (1) is reduced to the system

$$\partial_{\bar{z}}\Psi + B\bar{\Psi} = 0, \tag{2}$$

i.e. to system (1) with $A = 0$, by the gauge transform

$$\Psi \to \tilde{\Psi} = g^{-1}\Psi, \quad B \to \tilde{B} = g^{-1}B\bar{g}, \quad \partial_{\bar{z}}g + Ag = 0, \quad \det g \neq 0.$$

We say that the system

$$\partial_z \Psi^+ - \bar{\Psi}^+ B = 0 \tag{3}$$

is conjugate to system (2) (see [5] for a similar definition for $N = 1$).

We have the following result.

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Theorem 1 *Systems (2) and (3) are covariant, i.e. mapped into the systems of the same type, with respect to the Moutard type transform*

$$
\begin{aligned}
\Psi \to \tilde{\Psi} &= \Psi - F\,\omega_{F,F^+}^{-1}\,\omega_{\Psi,F^+},\\
\Psi^+ \to \tilde{\Psi}^+ &= \Psi^+ - \omega_{F,\Psi^+}\,\omega_{F,F^+}^{-1}\,F^+,\\
B \to \tilde{B} &= B + F\,\omega_{F,F^+}^{-1}\,F^+,
\end{aligned}
\tag{4}
$$

where F and F^+ are arbitrary fixed solutions of (2) and (3), respectively,

$$
\partial_{\bar z}\omega_{\Phi,\Phi^+} = \Phi^+\bar{\Phi}, \quad \operatorname{Re}\omega_{\Phi,\Phi^+} = 0,
\tag{5}
$$

for Φ and Φ^+ meeting equations (2) and (3), and $\det\omega_{F,F^+} \neq 0$*.*

For finding ω_{Φ,Φ^+} satisfying (5) we use also that $\partial_z\omega_{\Phi,\Phi^+} = -\bar{\Phi}^+\Phi$. In addition, our definition of ω_{Φ,Φ^+} is self-consistent up to a pure imaginary matrix integration constant in view of the identity $\partial_z\Phi^+\bar{\Phi} = -\partial_{\bar z}\bar{\Phi}^+\Phi$. The latter equality follows from systems (2) and (3) for Φ and Φ^+, respectively. We recall that the domain D is simply connected.

Given $\omega_{F,F^+}, \omega_{\Psi,F^+}$, and ω_{F,Ψ^+}, Theorem 1 is proved by straightforward computations.

In addition, for the system

$$
\partial_{\bar z}\Psi + A\Psi = 0,
\tag{6}
$$

i.e., for system (1) with $B = 0$, the following result also holds.

Proposition 1 *System (6) is covariant under the following Moutard type transform*

$$
\Psi \to \tilde{\Psi} = \Psi - F\,\hat{\omega}_{F,F^+}^{-1}\,\hat{\omega}_{\Psi,F^+}, \quad A \to \tilde{A} = A + F\,\hat{\omega}_{F,F^+}^{-1}\,F^+,
\tag{7}
$$

where F is an arbitrary fixed solution of (6), F^+ is an arbitrary fixed matrix function,

$$
\partial_{\bar z}\hat{\omega}_{\Phi,F^+} = F^+\Phi
\tag{8}
$$

for any matrix function Φ, and $\det\hat{\omega}_{F,F^+} \neq 0$*.*

Equations (7) and (8) are analogs of equations (4) and (5). However, in difference with (5), we do not require that the matrix functions $\hat{\omega}_{F,F^+}$ would be pure imaginary. Equation (8) is solvable for $\hat{\omega}_{F,F^+}$ and Proposition 1 is proved by straightforward computations.

Remark. Let $A, \tilde{A}, \Psi, F, F^+$, and $\hat{\omega}_{\Phi,F^+}$ be the same as in Proposition 1. Let

$$
g = 1 - F\hat{\omega}_{F,F^+}^{-1}\Lambda, \quad \Lambda_{\bar z} = \Lambda A + F^+.
$$

Then

$$
\partial_{\bar z}(g\Psi) + \tilde{A}(g\Psi) = 0.
$$

It is proved by straightforward computations and it shows that for invertible g the transform $A \to \tilde{A}$ reduces to a gauge transform.

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